

Instability of standing waves for a system of nonlinear Schrödinger equations in a degenerate case

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Abstract

We study a system of nonlinear Schrödinger equations with cubic interactions in one space dimension. The orbital stability and instability of semitrivial standing wave solutions are studied for both non-degenerate and degenerate cases.

Dedicated to Professor Nakao Hayashi on the occasion of his sixtieth birthday

1 Introduction

In this paper, we study the orbital stability and instability of standing wave solutions for the following system of nonlinear Schrödinger equations with cubic interactions in one space dimension:

$$\begin{cases} i\partial_t u_1 = -\partial_x^2 u_1 - \kappa_1 |u_1|^2 u_1 - \gamma u_2^2 \overline{u_1}, \\ i\partial_t u_2 = -\partial_x^2 u_2 - \kappa_2 |u_2|^2 u_2 - \gamma u_1^2 \overline{u_2}, \end{cases} \quad (1)$$

where u_1 and u_2 are complex-valued functions of $(t, x) \in \mathbb{R} \times \mathbb{R}$, and κ_1 , κ_2 and γ are positive constants. The system (1) appears in various areas of physics such as nonlinear optics, Bose-Einstein condensates, and so on (see, e.g., [1, 8, 9, 14]).

By the standard theory (see, e.g., [2, Chapter 4]), the Cauchy problem for (1) is globally well-posed in the energy space $H^1(\mathbb{R}, \mathbb{C})^2$, and the energy E and the charge Q are conserved, where

$$E(\vec{u}) = \sum_{j=1}^2 \left(\frac{1}{2} \|\partial_x u_j\|_{L^2}^2 - \frac{\kappa_j}{4} \|u_j\|_{L^4}^4 \right) - \frac{\gamma}{2} \operatorname{Re} \int_{\mathbb{R}} u_1^2 \overline{u_2}^2 dx,$$

$$Q(\vec{u}) = \frac{1}{2} \sum_{j=1}^2 \|u_j\|_{L^2}^2$$

for $\vec{u} := (u_1, u_2) \in H^1(\mathbb{R}, \mathbb{C})^2$. Note that (1) is written in a Hamiltonian form $i\partial_t \vec{u} = E'(\vec{u})$, and the conservation of charge follows from the invariance of E under gauge transform

$$E(e^{i\theta} \vec{u}) = E(e^{i\theta} u_1, e^{i\theta} u_2) = E(\vec{u})$$

for $\theta \in \mathbb{R}$ and $\vec{u} \in H^1(\mathbb{R}, \mathbb{C})^2$.

We study the orbital stability and instability of semitrivial standing wave solutions $e^{i\omega t} \vec{\phi}_\omega(x)$ for (1), where $\omega > 0$ is a constant,

$$\vec{\phi}_\omega(x) := \left(\frac{1}{\sqrt{\kappa_1}} \varphi_\omega(x), 0 \right), \quad (2)$$

and $\varphi_\omega(x) = \sqrt{2\omega} \operatorname{sech}(\sqrt{\omega} x)$ is a positive and even solution of

$$-\partial_x^2 \varphi + \omega \varphi - \varphi^3 = 0, \quad x \in \mathbb{R}.$$

We are mainly interested in the instability of $e^{i\omega t} \vec{\phi}_\omega(x)$ rather than the stability, and we assume the even symmetry for simplicity. We denote the set of even functions in $H^1(\mathbb{R})$ by $H_{\text{even}}^1(\mathbb{R})$, and define $X = H_{\text{even}}^1(\mathbb{R}, \mathbb{C})^2$. Note that $\varphi_\omega \in H_{\text{even}}^1(\mathbb{R})$ and $\vec{\phi}_\omega \in X$. Moreover, by the even symmetry of (1) and the uniqueness of solutions to the Cauchy problem for (1), if $\vec{u}_0 \in X$, then the solution $\vec{u}(t)$ of (1) with $\vec{u}(0) = \vec{u}_0$ satisfies $\vec{u} \in C(\mathbb{R}, X)$.

Definition 1. *We say that the standing wave solution $e^{i\omega t} \vec{\phi}_\omega$ of (1) is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If $u_0 \in X$ satisfies $\|\vec{u}_0 - \vec{\phi}_\omega\|_X < \delta$, then the solution $\vec{u}(t)$ of (1) with $\vec{u}(0) = \vec{u}_0$ satisfies*

$$\inf_{\theta \in \mathbb{R}} \|\vec{u}(t) - e^{i\theta} \vec{\phi}_\omega\|_X < \varepsilon$$

for all $t \in \mathbb{R}$. Otherwise, $e^{i\omega t} \vec{\phi}_\omega$ is called unstable.

We now state our main results in this paper.

Theorem 1. *Let $\kappa_1, \kappa_2, \gamma$ and ω be positive constants. Then, the semitrivial standing wave solution $e^{i\omega t} \vec{\phi}_\omega(x)$ of (1) is stable if $\gamma < \kappa_1$, and unstable if $\gamma > \kappa_1$.*

Theorem 2. *Let $\kappa_1, \kappa_2, \gamma$ and ω be positive constants, and let $\gamma = \kappa_1$. Then, the semitrivial standing wave solution $e^{i\omega t} \vec{\phi}_\omega(x)$ of (1) is stable if $\kappa_2 < \kappa_1$, and unstable if $\kappa_2 > \kappa_1$.*

Remark 1. By symmetry, similar results to Theorems 1 and 2 also hold for semitrivial standing wave solutions of the form

$$e^{i\omega t} \left(0, \frac{1}{\sqrt{\kappa_2}} \varphi_\omega(x) \right).$$

Remark 2. For the case $\gamma = \kappa_1 = \kappa_2$, the system (1) has an additional symmetry

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto R(\chi) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad R(\chi) := \begin{pmatrix} \cos \chi & -\sin \chi \\ \sin \chi & \cos \chi \end{pmatrix}$$

for $\chi \in \mathbb{R}$. By this symmetry, in the same way as in the proof of Theorem 1, we can prove that $e^{i\omega t} \vec{\phi}_\omega(x)$ is stable in the following weaker sense.

For any $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If $u_0 \in X$ satisfies $\|\vec{u}_0 - \vec{\phi}_\omega\|_X < \delta$, then the solution $\vec{u}(t)$ of (1) with $\vec{u}(0) = \vec{u}_0$ satisfies $\inf_{\theta, \chi \in \mathbb{R}} \|\vec{u}(t) - e^{i\theta} R(\chi) \vec{\phi}_\omega\|_X < \varepsilon$ for all $t \in \mathbb{R}$.

However, we do not know whether $e^{i\omega t} \vec{\phi}_\omega(x)$ is stable or not in the sense of Definition 1 for the case $\gamma = \kappa_1 = \kappa_2$.

Remark 3. The standing waves $e^{i\omega t} \vec{\phi}_\omega(x)$ are also solutions of the following system

$$\begin{cases} i\partial_t u_1 = -\partial_x^2 u_1 - \kappa_1 |u_1|^2 u_1 - \gamma |u_2|^2 u_1, \\ i\partial_t u_2 = -\partial_x^2 u_2 - \kappa_2 |u_2|^2 u_2 - \gamma |u_1|^2 u_2. \end{cases} \quad (3)$$

It is known that for any positive constants $\kappa_1, \kappa_2, \gamma$ and ω , the standing wave solution $e^{i\omega t} \vec{\phi}_\omega(x)$ is stable for (3) (see [12, 11]).

Remark 4. For related results on systems of nonlinear Schrödinger equations with quadratic interactions, see [5, 7]. While, for related studies on degenerate cases, see [10, 17].

The rest of the paper is organized as follows. In section 2, we consider the non-degenerate case $\gamma \neq \kappa_1$. The stability part of Theorem 1 is proved by the standard argument based on [6, 16]. The degenerate case $\gamma = \kappa_1$ is studied in section 3. The instability part of Theorem 2 is proved by using similar arguments to those in [5, 13].

2 Proof of Theorem 1

We regard $L^2(\mathbb{R}, \mathbb{C})$ as a real Hilbert space with the inner product

$$(u, v)_{L^2} = \operatorname{Re} \int_{\mathbb{R}} u(x) \overline{v(x)} dx,$$

and define the inner products of real Hilbert spaces $H = L^2_{\text{even}}(\mathbb{R}, \mathbb{C})^2$ and $X = H^1_{\text{even}}(\mathbb{R}, \mathbb{C})^2$ by

$$(\vec{u}, \vec{v})_H = (u_1, v_1)_{L^2} + (u_2, v_2)_{L^2}, \quad (\vec{u}, \vec{v})_X = (\vec{u}, \vec{v})_H + (\partial_x \vec{u}, \partial_x \vec{v})_H.$$

For $\omega > 0$, we define $S_\omega(\vec{v}) = E(\vec{v}) + \omega Q(\vec{v})$ for $\vec{v} \in X$. Then, we have $S'_\omega(\vec{\phi}_\omega) = 0$. Moreover, for $a \in \mathbb{R}$, we define L_a by

$$L_a u = -\partial_x^2 u + \omega u - a\varphi_\omega(x)^2 u$$

for $u \in H^1_{\text{even}}(\mathbb{R}, \mathbb{R})$. Then, for $\vec{v} = (v_1, v_2) \in X$, we have

$$\begin{aligned} \langle S''_\omega(\vec{\phi}_\omega) \vec{v}, \vec{v} \rangle &= \langle L_3 \text{Re } v_1, \text{Re } v_1 \rangle + \langle L_1 \text{Im } v_1, \text{Im } v_1 \rangle \\ &\quad + \langle L_{\gamma/\kappa_1} \text{Re } v_2, \text{Re } v_2 \rangle + \langle L_{-\gamma/\kappa_1} \text{Im } v_2, \text{Im } v_2 \rangle. \end{aligned} \quad (4)$$

We recall some known results on L_a (see [15]).

- Lemma 1.** (1) If $1 \leq a \leq 3$, then there exists $C > 0$ such that $\langle L_a v, v \rangle \geq C \|v\|_{H^1}^2$ for all $v \in H^1_{\text{even}}(\mathbb{R}, \mathbb{R})$ satisfying $(v, \varphi_\omega)_{L^2} = 0$.
(2) If $a < 1$, then there exists $C > 0$ such that $\langle L_a v, v \rangle \geq C \|v\|_{H^1}^2$ for all $v \in H^1_{\text{even}}(\mathbb{R}, \mathbb{R})$.
(3) $L_1 \varphi_\omega = 0$. If $a > 1$, then $\langle L_a \varphi_\omega, \varphi_\omega \rangle < 0$.

To prove the stability part of Theorem 1, we use the following proposition (see [6, 16]).

Proposition 2. Assume that there exists a constant $C > 0$ such that $\langle S''_\omega(\vec{\phi}_\omega) \vec{w}, \vec{w} \rangle \geq C \|\vec{w}\|_X^2$ for all $\vec{w} \in X$ satisfying

$$(\vec{w}, \vec{\phi}_\omega)_H = (\vec{w}, i\vec{\phi}_\omega)_H = 0. \quad (5)$$

Then, the standing wave solution $e^{i\omega t} \vec{\phi}_\omega$ of (1) is stable.

Proof of Theorem 1 (Stability part). Assume that $\gamma < \kappa_1$.

By Lemma 1 (1), there exists $C_1 > 0$ such that

$$\langle L_3 \text{Re } w_1, \text{Re } w_1 \rangle + \langle L_1 \text{Im } w_1, \text{Im } w_1 \rangle \geq C_1 \|w_1\|_{H^1}^2$$

for all $w_1 \in H^1(\mathbb{R}, \mathbb{C})$ satisfying

$$(\text{Re } w_1, \varphi_\omega)_{L^2} = (\text{Im } w_1, \varphi_\omega)_{L^2} = 0. \quad (6)$$

Note that since $\vec{\phi}_\omega$ has the form (2), the condition (6) is equivalent to (5). Moreover, by the assumption $0 < \gamma < \kappa_1$, we have $-\gamma/\kappa_1 < \gamma/\kappa_1 < 1$. Thus, by Lemma 1 (2), there exists $C_2 > 0$ such that

$$\langle L_{\gamma/\kappa_1} \text{Re } w_2, \text{Re } w_2 \rangle + \langle L_{-\gamma/\kappa_1} \text{Im } w_2, \text{Im } w_2 \rangle \geq C_2 \|w_2\|_{H^1}^2$$

for all $w_2 \in H_{\text{even}}^1(\mathbb{R}, \mathbb{C})$.

Thus, putting $C_3 = \min\{C_1, C_2\}$, we have $\langle S''_\omega(\vec{\phi}_\omega)\vec{w}, \vec{w} \rangle \geq C_3 \|\vec{w}\|_X^2$ for all $\vec{w} \in X$ satisfying (5). Hence, the stability part of Theorem 1 follows from Proposition 2. \square

Next, we consider the instability part of Theorem 1. The instability of $e^{i\omega t}\vec{\phi}_\omega$ can be proved for all $\gamma \in (\kappa_1, \infty)$ in the same way as in [3, 4] using the linear instability argument. On the other hand, by the Lyapunov function method, the instability of $e^{i\omega t}\vec{\phi}_\omega$ is proved for a restricted case $\gamma \in (\kappa_1, 3\kappa_1]$. Since our main interest in this paper is to consider the borderline case $\gamma = \kappa_1$ in Theorem 2, and since the instability result in Theorem 2 is proved by the Lyapunov function method but not by the linear instability argument, we here give the proof of instability for the case $\gamma \in (\kappa_1, 3\kappa_1]$. To prove the instability of $e^{i\omega t}\vec{\phi}_\omega$ for this case, we use the following proposition (see [13]).

Proposition 3. *Assume that there exist $\vec{\psi} \in X$ and a constant $C > 0$ such that*

$$(\vec{\psi}, \vec{\phi}_\omega)_H = (\vec{\psi}, i\vec{\phi}_\omega)_H = 0, \quad \langle S''_\omega(\vec{\phi}_\omega)\vec{\psi}, \vec{\psi} \rangle < 0,$$

and $\langle S''_\omega(\vec{\phi}_\omega)\vec{w}, \vec{w} \rangle \geq C \|\vec{w}\|_X^2$ for all $\vec{w} \in X$ satisfying

$$(\vec{w}, \vec{\phi}_\omega)_H = (\vec{w}, i\vec{\phi}_\omega)_H = (\vec{w}, \vec{\psi})_H = 0. \quad (7)$$

Then, the standing wave solution $e^{i\omega t}\vec{\phi}_\omega$ of (1) is unstable.

Proof of Theorem 1 (Instability part for the case $\kappa_1 < \gamma \leq 3\kappa_1$).

Assume that $\gamma \in (\kappa_1, 3\kappa_1]$. We take

$$\vec{\psi}_\omega = \left(0, \frac{1}{\sqrt{\kappa_1}}\varphi_\omega\right).$$

Then, $\vec{\psi}_\omega \in X$ and $(\vec{\psi}_\omega, \vec{\phi}_\omega)_H = (\vec{\psi}_\omega, i\vec{\phi}_\omega)_H = 0$.

Since $1 < \gamma/\kappa_1 \leq 3$, by Lemma 1 (3), we have

$$\langle S''_\omega(\vec{\phi}_\omega)\vec{\psi}_\omega, \vec{\psi}_\omega \rangle = \frac{1}{\kappa_1} \langle L_{\gamma/\kappa_1}\varphi_\omega, \varphi_\omega \rangle < 0.$$

Moreover, since the condition (7) is equivalent to

$$(\operatorname{Re} w_1, \varphi_\omega)_{L^2} = (\operatorname{Im} w_1, \varphi_\omega)_{L^2} = (\operatorname{Re} w_2, \varphi_\omega)_{L^2} = 0,$$

by Lemma 1 (1) and (2), there exists a constant $C > 0$ such that $\langle S''_\omega(\vec{\phi}_\omega)\vec{w}, \vec{w} \rangle \geq C \|\vec{w}\|_X^2$ for all $\vec{w} \in X$ satisfying (7).

Hence, the instability of $e^{i\omega t}\vec{\phi}_\omega$ follows from Proposition 3. \square

3 Proof of Theorem 2

In this section, we consider the case $\gamma = \kappa_1$. By (4), we have

$$\begin{aligned} \langle S''_\omega(\vec{\phi}_\omega)\vec{v}, \vec{v} \rangle &= \langle L_3 \operatorname{Re} v_1, \operatorname{Re} v_1 \rangle + \langle L_1 \operatorname{Im} v_1, \operatorname{Im} v_1 \rangle \\ &\quad + \langle L_1 \operatorname{Re} v_2, \operatorname{Re} v_2 \rangle + \langle L_{-1} \operatorname{Im} v_2, \operatorname{Im} v_2 \rangle \end{aligned} \quad (8)$$

for $\vec{v} = (v_1, v_2) \in X$. Recall that

$$\vec{\phi}_\omega = \left(\frac{1}{\sqrt{\kappa_1}} \varphi_\omega, 0 \right), \quad \vec{\psi}_\omega = \left(0, \frac{1}{\sqrt{\kappa_1}} \varphi_\omega \right).$$

Then, we have

$$\begin{aligned} \|\vec{\psi}_\omega\|_H &= \|\vec{\phi}_\omega\|_H, \quad (\vec{\psi}_\omega, \vec{\phi}_\omega)_H = (\vec{\psi}_\omega, i\vec{\phi}_\omega)_H = 0, \\ S''_\omega(\vec{\phi}_\omega)\vec{\psi}_\omega &= \left(0, \frac{1}{\sqrt{\kappa_1}} L_1 \varphi_\omega \right) = (0, 0), \\ S''_\omega(\vec{\phi}_\omega)\vec{\phi}_\omega &= S^{(3)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega) = \left(-\frac{2}{\sqrt{\kappa_1}} \varphi_\omega^3, 0 \right). \end{aligned} \quad (9)$$

In particular, we have

$$\langle S^{(3)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{\psi}_\omega \rangle = 0.$$

Moreover, we put

$$\begin{aligned} \nu_1 &:= \langle S^{(4)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega, \vec{\psi}_\omega), \vec{\psi}_\omega \rangle, \\ \nu_0 &:= \frac{1}{8} \langle S''_\omega(\vec{\phi}_\omega)\vec{\phi}_\omega, \vec{\phi}_\omega \rangle - \frac{1}{4} \langle S^{(3)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{\phi}_\omega \rangle + \frac{1}{4!} \nu_1. \end{aligned}$$

Then, by simple computations, we have

$$\nu_1 = -\frac{6\kappa_2}{\kappa_1^2} \|\varphi_\omega\|_{L^4}^4, \quad \nu_0 = \frac{\kappa_1 - \kappa_2}{4\kappa_1^2} \|\varphi_\omega\|_{L^4}^4. \quad (10)$$

As we will see below, the sign of ν_0 determines the stability and instability of $e^{i\omega t} \vec{\phi}_\omega$ for the borderline case $\gamma = \kappa_1$.

The following lemma plays an important role in the proof of Theorem 2 for both stability and instability results.

Lemma 4. *There exists a constant $k_0 > 0$ such that*

$$\langle S''_\omega(\phi_\omega)\vec{w}, \vec{w} \rangle \geq k_0 \|\vec{w}\|_X^2$$

for all $\vec{w} \in W$, where

$$W = \{\vec{w} \in X : (\vec{w}, \vec{\phi}_\omega)_H = (\vec{w}, i\vec{\phi}_\omega)_H = (\vec{w}, \vec{\psi}_\omega)_H = 0\}.$$

Proof. Since $\vec{w} \in W$ satisfies

$$(\operatorname{Re} w_1, \varphi_\omega)_{L^2} = (\operatorname{Im} w_1, \varphi_\omega)_{L^2} = (\operatorname{Re} w_2, \varphi_\omega)_{L^2} = 0,$$

the conclusion follows from (8) and Lemma 1. \square

Lemma 5. For $\lambda \in \mathbb{R}$,

$$S_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega) = S_\omega(\vec{\phi}_\omega) + \frac{\nu_1}{4!} \lambda^4, \quad \langle S'_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega), \vec{\psi}_\omega \rangle = \frac{\nu_1}{3!} \lambda^3.$$

Proof. By Taylor's expansion, we have

$$\begin{aligned} S_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega) &= S_\omega(\vec{\phi}_\omega) + \lambda \langle S'_\omega(\vec{\phi}_\omega), \vec{\psi}_\omega \rangle + \frac{\lambda^2}{2} \langle S''_\omega(\vec{\phi}_\omega) \vec{\psi}_\omega, \vec{\psi}_\omega \rangle \\ &\quad + \frac{\lambda^3}{3!} \langle S^{(3)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{\psi}_\omega \rangle + \frac{\lambda^4}{4!} \langle S^{(4)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega, \vec{\psi}_\omega), \vec{\psi}_\omega \rangle. \end{aligned}$$

Since $S'_\omega(\vec{\phi}_\omega) = S''_\omega(\vec{\phi}_\omega) \vec{\psi}_\omega = 0$ and $\langle S^{(3)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{\psi}_\omega \rangle = 0$, we have

$$S_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega) = S_\omega(\vec{\phi}_\omega) + \frac{\nu_1}{4!} \lambda^4.$$

Moreover, by differentiating this identity with respect to λ , we have the second identity. \square

Lemma 6. For $\lambda \in \mathbb{R}$,

$$S'_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega) = \frac{\lambda^2}{2} S^{(3)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega) + o(\lambda^2).$$

Proof. Since $S'_\omega(\vec{\phi}_\omega) = S''_\omega(\vec{\phi}_\omega) \vec{\psi}_\omega = 0$, we have

$$\begin{aligned} S'_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega) &= S'_\omega(\vec{\phi}_\omega) + \lambda S''_\omega(\vec{\phi}_\omega) \vec{\psi}_\omega + \frac{\lambda^2}{2} S^{(3)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega) + o(\lambda^2) \\ &= \frac{\lambda^2}{2} S^{(3)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega) + o(\lambda^2). \end{aligned}$$

This completes the proof. \square

Lemma 7. For $\lambda \in \mathbb{R}$ and $\vec{z} \in X$,

$$\begin{aligned} S_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega + \vec{z}) - S_\omega(\vec{\phi}_\omega) &= \frac{\nu_1}{4!} \lambda^4 + \frac{\lambda^2}{2} \langle S^{(3)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{z} \rangle + \frac{1}{2} \langle S''_\omega(\vec{\phi}_\omega) \vec{z}, \vec{z} \rangle \\ &\quad + o(\lambda^4 + \|\vec{z}\|_X^2). \end{aligned}$$

Proof. By Taylor's expansion, we have

$$\begin{aligned} S_\omega(\vec{\phi}_\omega + \lambda\vec{\psi}_\omega + \vec{z}) &= S_\omega(\vec{\phi}_\omega + \lambda\vec{\psi}_\omega) \\ &\quad + \langle S'_\omega(\vec{\phi}_\omega + \lambda\vec{\psi}_\omega), \vec{z} \rangle + \frac{1}{2} \langle S''_\omega(\vec{\phi}_\omega + \lambda\vec{\psi}_\omega) \vec{z}, \vec{z} \rangle + o(\|\vec{z}\|_X^2). \end{aligned}$$

Here, by Lemma 5, we have $S_\omega(\vec{\phi}_\omega + \lambda\vec{\psi}_\omega) = S_\omega(\vec{\phi}_\omega) + \frac{\nu_1}{4!} \lambda^4$.

Next, it follows from Lemma 6 that

$$\langle S'_\omega(\vec{\phi}_\omega + \lambda\vec{\psi}_\omega), \vec{z} \rangle = \frac{\lambda^2}{2} \langle S_\omega^{(3)}(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{z} \rangle + o(\lambda^4 + \|\vec{z}\|_X^2).$$

Moreover, we have

$$\langle S''_\omega(\vec{\phi}_\omega + \lambda\vec{\psi}_\omega) \vec{z}, \vec{z} \rangle = \langle S''_\omega(\vec{\phi}_\omega) \vec{z}, \vec{z} \rangle + O(\lambda \|\vec{z}\|_X^2).$$

Thus, we have the desired estimate. \square

Lemma 8. Let $\vec{v} = \lambda\vec{\psi}_\omega + \mu\vec{\phi}_\omega + \vec{w}$ with $\lambda, \mu \in \mathbb{R}$ and $\vec{w} \in W$. Assume that $\|\vec{\phi}_\omega + \vec{v}\|_H^2 = \|\vec{\phi}_\omega\|_H^2$. Then,

$$\mu = -\frac{\lambda^2}{2} + O(\mu^2 + \|\vec{w}\|_X^2).$$

Proof. Since $\vec{\psi}_\omega, \vec{\phi}_\omega$ and \vec{w} are orthogonal to each other in H , we have

$$\|\vec{\phi}_\omega\|_H^2 = \|\vec{\phi}_\omega + \vec{v}\|_H^2 = \lambda^2 \|\vec{\psi}_\omega\|_H^2 + (1 + \mu)^2 \|\vec{\phi}_\omega\|_H^2 + \|\vec{w}\|_H^2.$$

Moreover, since $\|\vec{\psi}_\omega\|_H = \|\vec{\phi}_\omega\|_H$, we have

$$\mu = -\frac{\lambda^2}{2} + \frac{1}{2} \left(\mu^2 + \frac{\|\vec{w}\|_H^2}{\|\vec{\phi}_\omega\|_H^2} \right),$$

which implies the desired result. \square

Lemma 9. Let $\vec{v} = \lambda\vec{\psi}_\omega + \mu\vec{\phi}_\omega + \vec{w}$ with $\lambda, \mu \in \mathbb{R}$ and $\vec{w} \in W$. Assume that $\|\vec{\phi}_\omega + \vec{v}\|_H^2 = \|\vec{\phi}_\omega\|_H^2$. Then,

$$E(\vec{\phi}_\omega + \vec{v}) - E(\vec{\phi}_\omega) = \nu_0 \lambda^4 + \frac{1}{2} \langle S''_\omega(\vec{\phi}_\omega) \vec{w}, \vec{w} \rangle + o(\lambda^4 + \|\vec{w}\|_X^2).$$

Proof. By Lemmas 7 and 8, we have

$$\begin{aligned}
E(\vec{\phi}_\omega + \vec{v}) - E(\vec{\phi}_\omega) &= S_\omega(\vec{\phi}_\omega + \vec{v}) - S_\omega(\vec{\phi}_\omega) \\
&= S_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega + \mu \vec{\phi}_\omega + \vec{w}) - S_\omega(\vec{\phi}_\omega) \\
&= \frac{\nu_1}{4!} \lambda^4 + \frac{\lambda^2}{2} \langle S_\omega^{(3)}(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \mu \vec{\phi}_\omega + \vec{w} \rangle \\
&\quad + \frac{1}{2} \langle S_\omega''(\vec{\phi}_\omega)(\mu \vec{\phi}_\omega + \vec{w}), \mu \vec{\phi}_\omega + \vec{w} \rangle + o(\lambda^4 + \|\mu \vec{\phi}_\omega + \vec{w}\|_X^2) \\
&= \nu_0 \lambda^4 + \frac{\lambda^2}{2} \langle S_\omega^{(3)}(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{w} \rangle - \frac{\lambda^2}{2} \langle S_\omega''(\vec{\phi}_\omega) \vec{\phi}_\omega, \vec{w} \rangle \\
&\quad + \frac{1}{2} \langle S_\omega''(\vec{\phi}_\omega) \vec{w}, \vec{w} \rangle + o(\lambda^4 + \|\vec{w}\|_X^2).
\end{aligned}$$

Here, by (9), the second and the third terms in the last equation cancel each other out. This completes the proof. \square

To prove the stability part of Theorem 2, we use the following proposition (see [6]). For $\varepsilon > 0$, we define

$$U_\varepsilon(\vec{\phi}_\omega) = \{\vec{u} \in X : \inf_{\theta \in \mathbb{R}} \|\vec{u} - e^{i\theta} \vec{\phi}_\omega\|_X < \varepsilon\}.$$

Proposition 10. *Assume that there exist positive constants p , C and ε such that*

$$E(\vec{u}) \geq E(\vec{\phi}_\omega) + C \inf_{\theta \in \mathbb{R}} \|\vec{u} - e^{i\theta} \vec{\phi}_\omega\|_X^p$$

for all $\vec{u} \in U_\varepsilon(\vec{\phi}_\omega)$ satisfying $Q(\vec{u}) = Q(\vec{\phi}_\omega)$. Then, the standing wave solution $e^{i\omega t} \vec{\phi}_\omega$ of (1) is stable.

Before proving the stability part of Theorem 2, we prepare one more lemma.

Lemma 11. *There exist $\varepsilon > 0$ and a C^2 -function $\alpha : U_\varepsilon(\vec{\phi}_\omega) \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ such that*

$$\begin{aligned}
\|\vec{u} - e^{i\alpha(\vec{u})} \vec{\phi}_\omega\|_H &\leq \|\vec{u} - e^{i\theta} \vec{\phi}_\omega\|_H, \quad \alpha(e^{i\theta} \vec{u}) = \alpha(\vec{u}) + \theta, \\
(\vec{u}, e^{i\alpha(\vec{u})} i \vec{\phi}_\omega)_H &= 0, \quad i\alpha'(\vec{u}) = -\frac{e^{i\alpha(\vec{u})} \vec{\phi}_\omega}{(\vec{u}, e^{i\alpha(\vec{u})} \vec{\phi}_\omega)_H}
\end{aligned} \tag{11}$$

for all $\vec{u} \in U_\varepsilon(\vec{\phi}_\omega)$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Proof. See Lemma 3.2 of [6]. \square

Proof of Theorem 2 (Stability part). Assume that $\gamma = \kappa_1 > \kappa_2$.

Let $\vec{u} \in U_\varepsilon(\vec{\phi}_\omega)$ with $Q(\vec{u}) = Q(\vec{\phi}_\omega)$. Then, for $\alpha(\vec{u})$ given in Lemma 11, we have

$$\|\vec{u} - e^{i\alpha(\vec{u})}\vec{\phi}_\omega\|_X \leq \left(1 + \frac{2\|\vec{\phi}_\omega\|_X}{\|\vec{\phi}_\omega\|_H}\right)\varepsilon. \quad (12)$$

Indeed, for $\beta(\vec{u}) \in \mathbb{R}$ such that

$$\|\vec{u} - e^{i\beta(\vec{u})}\vec{\phi}_\omega\|_X = \inf_{\theta \in \mathbb{R}} \|\vec{u} - e^{i\theta}\vec{\phi}_\omega\|_X < \varepsilon,$$

we have

$$\begin{aligned} |e^{i\alpha(\vec{u})} - e^{i\beta(\vec{u})}| \|\vec{\phi}_\omega\|_H &\leq \|e^{i\alpha(\vec{u})}\vec{\phi}_\omega - \vec{u}\|_H + \|\vec{u} - e^{i\beta(\vec{u})}\vec{\phi}_\omega\|_H \\ &\leq 2\|\vec{u} - e^{i\beta(\vec{u})}\vec{\phi}_\omega\|_H \leq 2\|\vec{u} - e^{i\beta(\vec{u})}\vec{\phi}_\omega\|_X < 2\varepsilon, \end{aligned}$$

and $\|\vec{u} - e^{i\alpha(\vec{u})}\vec{\phi}_\omega\|_X \leq \|\vec{u} - e^{i\beta(\vec{u})}\vec{\phi}_\omega\|_X + |e^{i\alpha(\vec{u})} - e^{i\beta(\vec{u})}| \|\vec{\phi}_\omega\|_X$, which implies (12).

Let $\vec{v} = e^{-i\alpha(\vec{u})}\vec{u} - \vec{\phi}_\omega$. Then, we have $(\vec{v}, i\vec{\phi}_\omega)_H = 0$, and we decompose \vec{v} as $\vec{v} = \lambda\vec{\psi}_\omega + \mu\vec{\phi}_\omega + \vec{w}$ with $\lambda, \mu \in \mathbb{R}$ and $\vec{w} \in W$.

Since $\|\vec{\phi}_\omega + \vec{v}\|_H^2 = \|\vec{u}\|_H^2 = 2Q(\vec{u}) = 2Q(\vec{\phi}_\omega) = \|\vec{\phi}_\omega\|_H^2$, it follows from Lemmas 9 and 4 that

$$E(\vec{u}) - E(\vec{\phi}_\omega) \geq \nu_0\lambda^4 + \frac{k_0}{2}\|\vec{w}\|_X^2 + o(\lambda^4 + \|\vec{w}\|_X^2).$$

Here, we note that k_0 is the positive constant given in Lemma 4, and that $\nu_0 > 0$ by (10) and the assumption $\kappa_1 > \kappa_2$.

Moreover, by Lemma 8, we have

$$\begin{aligned} \inf_{\theta \in \mathbb{R}} \|\vec{u} - e^{i\theta}\vec{\phi}_\omega\|_X &\leq \|\vec{v}\|_X \leq |\lambda|\|\vec{\psi}_\omega\|_X + |\mu|\|\vec{\phi}_\omega\|_X + \|\vec{w}\|_X \\ &= |\lambda|\|\vec{\psi}_\omega\|_X + \|\vec{w}\|_X + O(\lambda^2 + \|\vec{w}\|_X^2). \end{aligned}$$

Thus, taking ε smaller if necessary, we have

$$E(\vec{u}) - E(\vec{\phi}_\omega) \geq \frac{\nu_0}{2}\lambda^4 + \frac{k_0}{4}\|\vec{w}\|_X^2 \geq C_1 \inf_{\theta \in \mathbb{R}} \|\vec{u} - e^{i\theta}\vec{\phi}_\omega\|_X^4$$

for some $C_1 > 0$.

Hence, the stability of $e^{i\omega t}\vec{\phi}_\omega$ follows from Proposition 10. \square

In the rest of this section, we study the instability of $e^{i\omega t}\vec{\phi}_\omega$ for the case $\gamma = \kappa_1 < \kappa_2$. We follow the argument used in [5, 13].

For $\vec{u} \in U_\varepsilon(\vec{\phi}_\omega)$, we define

$$\begin{aligned} M(\vec{u}) &= e^{-i\alpha(\vec{u})}\vec{u}, \quad A(\vec{u}) = (iM(\vec{u}), \vec{\psi}_\omega)_H, \\ q(\vec{u}) &= e^{i\alpha(\vec{u})}\vec{\psi}_\omega + (M(\vec{u}), \vec{\psi}_\omega)_H i\alpha'(\vec{u}), \\ P(\vec{u}) &= \langle E'(\vec{u}), q(\vec{u}) \rangle. \end{aligned} \tag{13}$$

Then, we have the following lemmas (see [6]).

Lemma 12. For $\vec{u} \in U_\varepsilon(\vec{\phi}_\omega)$,

- (1) $A(e^{i\theta}\vec{u}) = A(\vec{u})$, $q(e^{i\theta}\vec{u}) = e^{i\theta}q(\vec{u})$ for all $\theta \in \mathbb{R}$.
- (2) $\langle A'(\vec{u}), \vec{w} \rangle = (q(\vec{u}), i\vec{w})_H$ for $\vec{w} \in X$.
- (3) $q(\vec{\phi}_\omega) = \vec{\psi}_\omega$, $\langle Q'(\vec{u}), q(\vec{u}) \rangle = 0$.

Lemma 13. Let I be an interval of \mathbb{R} . Let $\vec{u} \in C(I, X)$ be a solution of (1), and assume that $\vec{u}(t) \in U_\varepsilon(\vec{\phi}_\omega)$ for all $t \in I$. Then,

$$\frac{d}{dt}A(\vec{u}(t)) = P(\vec{u}(t)) \quad \text{for all } t \in I.$$

By Lemma 12 and (11), we have

$$\begin{aligned} P(\vec{u}) &= \langle S'_\omega(\vec{u}), q(\vec{u}) \rangle \\ &= \langle S'_\omega(M(\vec{u})), \vec{\psi}_\omega \rangle - \frac{(M(\vec{u}), \vec{\psi}_\omega)_H}{(M(\vec{u}), \vec{\phi}_\omega)_H} \langle S'_\omega(M(\vec{u})), \vec{\phi}_\omega \rangle. \end{aligned} \tag{14}$$

We prove the following.

Proposition 14. Let $\gamma = \kappa_1 < \kappa_2$. Then, there exists a constant $\varepsilon_0 > 0$ such that

$$E(\vec{\phi}_\omega) \leq E(\vec{u}) - \frac{(M(\vec{u}), \vec{\psi}_\omega)_H}{2\|\vec{\psi}_\omega\|_H^2} P(\vec{u})$$

for all $\vec{u} \in U_{\varepsilon_0}(\vec{\phi}_\omega)$ satisfying $Q(\vec{u}) = Q(\vec{\phi}_\omega)$.

For the proof of Proposition 14, we prove several lemmas.

Lemma 15. For $\lambda \in \mathbb{R}$ and $\vec{z} \in X$,

$$\begin{aligned} &\lambda \langle S'_\omega(\vec{\phi}_\omega + \lambda\vec{\psi}_\omega + \vec{z}), \vec{\psi}_\omega \rangle \\ &= \frac{\nu_1}{3!}\lambda^4 + \lambda^2 \langle S_\omega^{(3)}(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{z} \rangle + o(\lambda^4 + \|\vec{z}\|_X^2). \end{aligned}$$

Proof. By Taylor's expansion, we have

$$\begin{aligned} & \lambda \langle S'_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega + \vec{z}), \vec{\psi}_\omega \rangle \\ &= \lambda \langle S'_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega), \vec{\psi}_\omega \rangle + \lambda \langle S''_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega) \vec{z}, \vec{\psi}_\omega \rangle + O(\lambda \|\vec{z}\|_X^2). \end{aligned}$$

Here, by Lemma 5, we have $\lambda \langle S'_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega), \vec{\psi}_\omega \rangle = \frac{\nu_1}{3!} \lambda^4$.

Next, since $S''_\omega(\vec{\phi}_\omega) \vec{\psi}_\omega = 0$, we have

$$\begin{aligned} & \lambda \langle S''_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega) \vec{z}, \vec{\psi}_\omega \rangle \\ &= \lambda \langle S''_\omega(\vec{\phi}_\omega) \vec{z}, \vec{\psi}_\omega \rangle + \lambda^2 \langle S^{(3)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{z}), \vec{\psi}_\omega \rangle + o(\lambda^2 \|\vec{z}\|_X) \\ &= \lambda^2 \langle S^{(3)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{z} \rangle + o(\lambda^2 \|\vec{z}\|_X). \end{aligned}$$

Thus, we obtain the desired result. \square

Lemma 16. For $\lambda \in \mathbb{R}$ and $\vec{z} \in X$,

$$\begin{aligned} & \lambda^2 \langle S'_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega + \vec{z}), \vec{\phi}_\omega \rangle \\ &= \frac{\lambda^4}{2} \langle S^{(3)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{\phi}_\omega \rangle + \lambda^2 \langle S''_\omega(\vec{\phi}_\omega) \vec{\phi}_\omega, \vec{z} \rangle + o(\lambda^4 + \|\vec{z}\|_X^2). \end{aligned}$$

Proof. By Taylor's expansion, we have

$$\begin{aligned} & \langle S'_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega + \vec{z}), \vec{\phi}_\omega \rangle \\ &= \langle S'_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega), \vec{\phi}_\omega \rangle + \langle S''_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega) \vec{z}, \vec{\phi}_\omega \rangle + O(\|\vec{z}\|_X^2). \end{aligned}$$

Here, it follows from Lemma 6 that

$$\langle S'_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega), \vec{\phi}_\omega \rangle = \frac{\lambda^2}{2} \langle S^{(3)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{\phi}_\omega \rangle + o(\lambda^2).$$

Moreover, we have $\langle S''_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega) \vec{z}, \vec{\phi}_\omega \rangle = \langle S''_\omega(\vec{\phi}_\omega) \vec{z}, \vec{\phi}_\omega \rangle + O(\lambda \|\vec{z}\|_X)$.

Thus, we have

$$\begin{aligned} \langle S'_\omega(\vec{\phi}_\omega + \lambda \vec{\psi}_\omega + \vec{z}), \vec{\phi}_\omega \rangle &= \frac{\lambda^2}{2} \langle S^{(3)}_\omega(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{\phi}_\omega \rangle \\ &\quad + \langle S''_\omega(\vec{\phi}_\omega) \vec{\phi}_\omega, \vec{z} \rangle + o(\lambda^2) + O(\lambda \|\vec{z}\|_X) + O(\|\vec{z}\|_X^2), \end{aligned}$$

which implies the desired result. \square

Lemma 17. Let $\vec{v} = \lambda \vec{\psi}_\omega + \mu \vec{\phi}_\omega + \vec{w}$ with $\lambda, \mu \in \mathbb{R}$ and $\vec{w} \in W$. Assume that $\|\vec{\phi}_\omega + \vec{v}\|_H^2 = \|\vec{\phi}_\omega\|_H^2$. Then,

$$\lambda \langle S'_\omega(\vec{\phi}_\omega + \vec{v}), \vec{\psi}_\omega \rangle - \lambda^2 \langle S'_\omega(\vec{\phi}_\omega + \vec{v}), \vec{\phi}_\omega \rangle = 4\nu_0 \lambda^4 + o(\lambda^4 + \|\vec{w}\|_X^2).$$

Proof. By Lemmas 8 and 15, we have

$$\begin{aligned}\lambda\langle S'_\omega(\vec{\phi}_\omega + \vec{v}), \vec{\psi}_\omega \rangle &= \lambda\langle S'_\omega(\vec{\phi}_\omega + \lambda\vec{\psi}_\omega + \mu\vec{\phi}_\omega + \vec{w}), \vec{\psi}_\omega \rangle \\ &= \frac{\nu_1}{3!}\lambda^4 - \frac{\lambda^4}{2}\langle S_\omega^{(3)}(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{\phi}_\omega \rangle \\ &\quad + \lambda^2\langle S_\omega^{(3)}(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{w} \rangle + o(\lambda^4 + \|\vec{w}\|_X^2).\end{aligned}$$

On the other hand, by Lemmas 8 and 16, we have

$$\begin{aligned}\lambda^2\langle S'_\omega(\vec{\phi}_\omega + \vec{v}), \vec{\phi}_\omega \rangle &= \lambda^2\langle S'_\omega(\vec{\phi}_\omega + \lambda\vec{\psi}_\omega + \mu\vec{\phi}_\omega + \vec{w}), \vec{\phi}_\omega \rangle \\ &= \frac{\lambda^4}{2}\langle S_\omega^{(3)}(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{\phi}_\omega \rangle + \lambda^2\mu\langle S_\omega''(\vec{\phi}_\omega)\vec{\phi}_\omega, \vec{\phi}_\omega \rangle \\ &\quad + \lambda^2\langle S_\omega''(\vec{\phi}_\omega)\vec{\phi}_\omega, \vec{w} \rangle + o(\lambda^4 + \|\mu\vec{\phi}_\omega + \vec{w}\|_X^2) \\ &= \frac{\lambda^4}{2}\langle S_\omega^{(3)}(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{\phi}_\omega \rangle - \frac{\lambda^4}{2}\langle S_\omega''(\vec{\phi}_\omega)\vec{\phi}_\omega, \vec{\phi}_\omega \rangle \\ &\quad + \lambda^2\langle S_\omega''(\vec{\phi}_\omega)\vec{\phi}_\omega, \vec{w} \rangle + o(\lambda^4 + \|\vec{w}\|_X^2).\end{aligned}$$

Thus, we have

$$\begin{aligned}\lambda\langle S'_\omega(\vec{\phi}_\omega + \vec{v}), \vec{\psi}_\omega \rangle - \lambda^2\langle S'_\omega(\vec{\phi}_\omega + \vec{v}), \vec{\phi}_\omega \rangle \\ &= 4\nu_0\lambda^4 - \lambda^2\langle S_\omega''(\vec{\phi}_\omega)\vec{\phi}_\omega, \vec{w} \rangle + \lambda^2\langle S_\omega^{(3)}(\vec{\phi}_\omega)(\vec{\psi}_\omega, \vec{\psi}_\omega), \vec{w} \rangle \\ &\quad + o(\lambda^4 + \|\vec{w}\|_X^2).\end{aligned}$$

Finally, by (9), we obtain the desired result. \square

We are now in a position to give the proof of Proposition 14.

Proof of Proposition 14. Let $\vec{u} \in U_\varepsilon(\vec{\phi}_\omega)$ with $Q(\vec{u}) = Q(\vec{\phi}_\omega)$. We put $\vec{v} = M(\vec{u}) - \vec{\phi}_\omega$, and decompose \vec{v} as $\vec{v} = \lambda\vec{\psi}_\omega + \mu\vec{\phi}_\omega + \vec{w}$ with $\lambda, \mu \in \mathbb{R}$ and $\vec{w} \in W$. Here, we note that $(\vec{v}, i\vec{\phi}_\omega)_H = 0$ by Lemma 11,

$$\lambda = \frac{(\vec{v}, \vec{\psi}_\omega)_H}{\|\vec{\psi}_\omega\|_H^2} = \frac{(M(\vec{u}), \vec{\psi}_\omega)_H}{\|\vec{\psi}_\omega\|_H^2},$$

and $\|\vec{v}\|_X \leq C\varepsilon$ for some $C > 0$ by (12).

Since $\|\vec{\phi}_\omega + \vec{v}\|_H^2 = \|\vec{u}\|_H^2 = 2Q(\vec{u}) = 2Q(\vec{\phi}_\omega) = \|\vec{\phi}_\omega\|_H^2$, it follows from Lemmas 9 and 4 that

$$E(\vec{u}) - E(\vec{\phi}_\omega) \geq \nu_0\lambda^4 + \frac{k_0}{2}\|\vec{w}\|_X^2 + o(\lambda^4 + \|\vec{w}\|_X^2).$$

Moreover, by (14) and Lemmas 8, 16 and 17, we have

$$\begin{aligned}
\lambda P(\vec{u}) &= \lambda \langle S'_\omega(\vec{\phi}_\omega + \vec{v}), \vec{\psi}_\omega \rangle - \lambda \frac{(\vec{\phi}_\omega + \vec{v}, \vec{\psi}_\omega)_H}{(\vec{\phi}_\omega + \vec{v}, \vec{\phi}_\omega)_H} \langle S'_\omega(\vec{\phi}_\omega + \vec{v}), \vec{\phi}_\omega \rangle \\
&= \lambda \langle S'_\omega(\vec{\phi}_\omega + \vec{v}), \vec{\psi}_\omega \rangle - \frac{\lambda^2 \|\vec{\psi}_\omega\|_H^2}{(1 + \mu) \|\vec{\phi}_\omega\|_H^2} \langle S'_\omega(\vec{\phi}_\omega + \vec{v}), \vec{\phi}_\omega \rangle \\
&= \lambda \langle S'_\omega(\vec{\psi}_\omega + \vec{v}), \vec{\psi}_\omega \rangle - \lambda^2 \langle S'_\omega(\vec{\phi}_\omega + \vec{v}), \vec{\phi}_\omega \rangle + o(\lambda^4 + \|\vec{w}\|_X^2) \\
&= 4\nu_0 \lambda^4 + o(\lambda^4 + \|\vec{w}\|_X^2).
\end{aligned}$$

Here, we used the fact that $\|\vec{\psi}_\omega\|_H = \|\vec{\phi}_\omega\|_H$. Thus, we have

$$E(\vec{u}) - E(\vec{\phi}_\omega) - \frac{\lambda}{2} P(\vec{u}) \geq -\nu_0 \lambda^4 + \frac{k_0}{2} \|\vec{w}\|_X^2 + o(\lambda^4 + \|\vec{w}\|_X^2).$$

Since $k_0 > 0$ and $-\nu_0 > 0$ by (10) and the assumption $\kappa_1 < \kappa_2$, taking ε smaller if necessary, we have

$$E(\vec{u}) - E(\vec{\phi}_\omega) \geq \frac{\lambda}{2} P(\vec{u}) = \frac{(M(\vec{u}), \vec{\psi}_\omega)_H}{2 \|\vec{\psi}_\omega\|_H^2} P(\vec{u}).$$

This completes the proof. \square

Finally, we prove the instability part of Theorem 2.

Proof of Theorem 2 (Instability part). Suppose that $e^{i\omega t} \vec{\phi}_\omega$ is stable. For λ close to 0, we define

$$\vec{\varphi}_\lambda = \vec{\phi}_\omega + \lambda \vec{\psi}_\omega + \sigma(\lambda) \vec{\phi}_\omega, \quad \sigma(\lambda) = (1 - \lambda^2)^{1/2} - 1.$$

Then, we have $Q(\vec{\varphi}_\lambda) = Q(\vec{\phi}_\omega)$. Moreover, since $\nu_0 < 0$, by Lemma 9, there exists $\lambda_1 > 0$ such that

$$\delta_\lambda := E(\vec{\phi}_\omega) - E(\vec{\varphi}_\lambda) = -\nu_0 \lambda^4 + o(\lambda^4) > 0$$

for $\lambda \in (-\lambda_1, 0) \cup (0, \lambda_1)$.

Let $\vec{u}_\lambda(t)$ be the solution of (1) with $\vec{u}_\lambda(0) = \vec{\varphi}_\lambda$. Since $e^{i\omega t} \vec{\phi}_\omega$ is stable, there exists $\lambda_0 \in (0, \lambda_1)$ such that if $|\lambda| < \lambda_0$, then $\vec{u}_\lambda(t) \in U_{\varepsilon_0}(\vec{\phi}_\omega)$ for all $t \geq 0$, where ε_0 is the positive constant given in Proposition 14.

By the definition (13) of M and A , there exist positive constants C_1 and C_2 such that

$$|M(\vec{v})| \leq C_1 \|\vec{\psi}_\omega\|_H, \quad |A(\vec{v})| \leq C_2$$

for all $\vec{v} \in U_{\varepsilon_0}(\vec{\phi}_\omega)$.

For $\lambda \in (-\lambda_0, 0) \cup (0, \lambda_0)$, by Proposition 14 and the conservation of E and Q , we have

$$0 < \delta_\lambda = E(\vec{\phi}_\omega) - E(\vec{u}_\lambda(t)) \leq C_1 |P(\vec{u}_\lambda(t))|$$

for all $t \geq 0$. Since $t \mapsto P(\vec{u}_\lambda(t))$ is continuous, we see that either (i) $P(\vec{u}_\lambda(t)) \geq \delta_\lambda/C_1$ for all $t \geq 0$, or (ii) $P(\vec{u}_\lambda(t)) \leq -\delta_\lambda/C_1$ for all $t \geq 0$. Moreover, by Lemma 13, we have

$$\frac{d}{dt}A(\vec{u}_\lambda(t)) = P(\vec{u}_\lambda(t))$$

for all $t \geq 0$. Therefore, we see that $|A(\vec{u}_\lambda(t))| \rightarrow \infty$ as $t \rightarrow \infty$. This contradicts the fact that $|A(\vec{u}_\lambda(t))| \leq C_2$ for all $t \geq 0$. Hence, $e^{i\omega t}\vec{\phi}_\omega$ is unstable. \square

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